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THE GAUSSIAN COTYPE OF OPERATORS FROM $C(K)$

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ABSTRACT

We show that the canonical embedding $C(K) \to L_\Phi(\mu)$ has Gaussian cotype p, where μ is a Radon probability measure on K, and Φ is an Orlicz function equivalent to $t^p(\log t)^{p/2}$ for large t.

In [7], I showed that the Gaussian cotype 2 constant of the canonical embedding $l_\infty^N \to L_{2,1}^N$ is bounded by $\log \log N$. Talagrand [9] showed that this embedding does not have uniformly bounded cotype 2 constant. In fact, a careful study of his proof yields that the cotype 2 constant is bounded below by $\sqrt{\log \log N}$. In this paper, we will show that this is the correct value for the Gaussian cotype 2 constant of this operator. However, we will show this via a different result, which we will give presently. First, let us define our terms.

We will write Φ_p for an Orlicz function such that $\Phi_p(t) \approx t^p(\log t)^{p/2}$ for large t.

For any bounded linear operator $T: X \to Y$, where X and Y are Banach spaces, and any $2 \leqq p < \infty$, we say that T has *Gaussian cotype p* if there is a number $C < \infty$ such that for all sequences $x_1, x_2, \ldots \in X$ we have

$$\mathbf{E} \left\| \sum_{s=1}^{\infty} \gamma_s x_s \right\| \geqq C^{-1} \left(\sum_{s=1}^{\infty} \| Tx_s \|^p \right)^{1/p}.$$

(Here, as elsewhere, $\gamma_1, \gamma_2, \ldots$ denote independent $N(0, 1)$ Gaussian random variables.) We call the least value of C the *Gaussian cotype p constant* of T, and denote it by $\beta^{(p)}(T)$.

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Throughout this paper, we shall use the letter c to denote a positive finite constant, whose value may change with each occurrence. We shall write $A \approx B$ to mean $A \leqq cB$ and $B \leqq cA$.

THEOREM 1. *Let μ be a Radon probability measure on a compact Hausdorff topological space K, and let $2 \leqq p < \infty$. Then the canonical embedding $C(K) \to L_{\Phi_p}(\mu)$ has Gaussian cotype p.*

Finding the Gaussian cotype p constant of an operator from $C(K)$ involves finding lower bounds for the quantity $\mathbf{E} \| \Sigma_{s=1}^{\infty} \gamma_s x_s \|_\infty$, where $x_1, x_2, \ldots \in C(K)$. In fact, since we really only need to consider finite sequences $x_1, x_2, \ldots, x_S \in C(K)$, in order to prove Theorem 1, it is sufficient to show that the Gaussian cotype p constant of the canonical embedding $C(K) \to L_{\Phi_p}(\mu)$ is uniformly bounded over all *finite* K. Now we see that we are trying to find lower bounds for the supremum of the finite Gaussian process, $\sup_{\omega \in K} |\Gamma_\omega|$, where $\Gamma_\omega = \Sigma_{s=1}^{\infty} \gamma_s x_s(\omega)$. Hence we can apply the following result due to Talagrand [8].

THEOREM 2. *Let $(\Gamma_\omega : \omega \in K)$ be a finite Gaussian process.*

(i) *Let*

$$V_1 = \mathbf{E}\left(\sup_{\omega \in K} |\Gamma_\omega| \right).$$

(ii) *Let V_2 be the infimum of*

$$\left(\sup_{t \geqq 1} \sqrt{1 + \log t}\, (\mathbf{E} |Y_t|^2)^{1/2} \right) \left(\sup_{\omega \in K} \sum_{t=1}^{\infty} |\alpha_t(\omega)| \right)$$

over all Gaussian processes $(Y_t)_{t=1}^{\infty}$ and over all sequences $(\alpha_t)_{t=1}^{\infty}$ of functions on K such that $\Gamma_\omega = \Sigma_{t=1}^{\infty} \alpha_t(\omega) Y_t$.

Then $V_1 \approx V_2$.

We can rewrite this theorem in the following way. First, let us define the following spaces (here we are assuming K is finite).

$$\mathcal{G} = \left\{ (x_s \in C(K))_{s=1}^{\infty} : \| (x_s) \|_{\mathcal{G}} = \mathbf{E} \left\| \sum_{s=1}^{\infty} \gamma_s x_s \right\|_\infty < \infty \right\},$$

$$C(K, l_1) = \left\{ (\alpha_t \in C(K))_{t=1}^{\infty} : \| (\alpha_t) \|_{C(K,l_1)} = \left\| \sum_{t=1}^{\infty} |\alpha_t| \right\|_\infty < \infty \right\},$$

$$\mathcal{Y} = \left\{ (y_t \in l_2)_{t=1}^{\infty} : \| (y_t) \|_{\mathcal{Y}} = \sup_{t \geqq 1} \sqrt{1 + \log t}\, \| y_t \|_2 < \infty \right\}.$$

Let $m : C(K, l_1) \times \mathcal{Y} \to \mathcal{G}$ be the bilinear map $m((\alpha_t), (y_t)) = (x_s)$, where

$$x_s = \sum_{t=1}^{\infty} y_t(s)\alpha_t.$$

COROLLARY 3. *The map m has the following two properties*:

(i) *m is bounded*;

(ii) *m is open, that is, if* $\| (x_s) \|_{\mathcal{G}} \leqq 1$, *then there are* $\| (\alpha_t) \|_{C(K,l_1)} \leqq c$ *and* $\| (y_t) \|_{\mathcal{Y}} \leqq c$ *such that* $m((\alpha_t), (y_t)) = (x_s)$.

PROOF. This is just restating Theorem 2, setting $\Gamma_\omega = \Sigma_{s=1}^{\infty} \gamma_s x_s(\omega)$, and $Y_t = \Sigma_{s=1}^{\infty} \gamma_s y_t(s)$. □

From this we obtain the following corollary, for which we first give a definition.

DEFINITION. If $2 \leqq p < \infty$, and $T: C(K) \to Y$ is a bounded linear map, where K is a finite Hausdorff space, and Y is a Banach space, then we set

$$H^{(p)}(T) = \sup \left\{ \left(\sum_{s=1}^{\infty} \| Tx_s \|^p \right)^{1/p} \right\},$$

where the supremum is over all $x_s = \Sigma_{t=1}^{\infty} y_t(s)\alpha_t$, with $\alpha_1, \alpha_2, \ldots$ pairwise disjoint elements of the unit ball of $C(K)$, and $\| (y_t) \|_2 \leqq 1/\sqrt{1 + \log t}$ for each $t \geqq 1$.

COROLLARY 4. *For any* $2 \leqq p < \infty$, *and for any bounded linear operator* $T: C(K) \to Y$, *where K is a finite Hausdorff space, and Y is a Banach space, we have*

$$H^{(p)}(T) \approx \beta^{(p)}(T).$$

PROOF. This follows straight away from Corollary 3 and the following lemma.

LEMMA 5. *Let B be the set of* $(\alpha_t) \in C(K, l_1)$ *such that the* α_t *are pairwise disjoint elements of the unit ball of* $C(K)$. *Then the closed convex hull of B is the unit ball of* $C(K, l_1)$.

PROOF. See [5], Lemma 4 or [3], Proposition 14.4. □

Now we are almost in a position to prove Theorem 1; we just need the following properties of $L_{\Phi_p}(\mu)$ (see [4]).

LEMMA 6. *If μ is a Radon probability measure on a compact Hausdorff space K, then*

(i) *for any Borel subset I of K, we have* $\| \chi_I \|_{\Phi_p} \approx (\mu(I))^{1/p} \sqrt{\log(1/\mu(I))}$;

(ii) *the space L_{Φ_p} satisfies an upper p estimate.*

PROOF OF THEOREM 1. We want to show that $H^{(p)}(C(K) \to L_{\Phi_p}(\mu)) \leqq c$, where μ is a probability measure on a finite Hausdorff space K. So consider (x_s), (α_t) and (y_t) as given in the definition of $H^{(p)}(T)$. Then we need to show that

$$\sum_{s=1}^{\infty} \| x_s \|_{\Phi_p}^p \leqq c.$$

First note, by Lemma 6, that

$$\| x_s \|_{\Phi_p}^p \leqq c \sum_{t=1}^{\infty} | y_t(s) |^p \| \alpha_t \|_{\Phi_p}^p$$

$$\leqq c \sum_{t=1}^{\infty} | y_t(s) |^p \mu(I_t) \left(\log \frac{1}{\mu(I_t)} \right)^{p/2},$$

where I_t is the support of α_t. Hence

$$\sum_{s=1}^{\infty} \| x_s \|_{\Phi_p}^p \leqq c \sum_{t=1}^{\infty} \sum_{s=1}^{\infty} | y_t(s) |^p \mu(I_t) \left(\log \frac{1}{\mu(I_t)} \right)^{p/2}$$

$$\leqq c \sum_{t=1}^{\infty} \frac{1}{(1 + \log t)^{p/2}} \mu(I_t) \left(\log \frac{1}{\mu(I_t)} \right)^{p/2},$$

since $\| y_t \|_p \leqq \| y_t \|_2 \leqq 1/\sqrt{1 + \log t}$. But now, splitting the sum into the two cases $\mu(I_t) \geqq 1/t^2$ or $\mu(I_t) < 1/t^2$, we deduce that this sum is bounded by some universal constant. □

Concluding remarks

We first note that there is a nice way to calculate the Orlicz norms $\| \cdot \|_{\Phi_p}$ provided by the following result of Bennett and Rudnick.

THEOREM 7. *If $1 \leqq p < \infty$ and $a \in \mathbf{R}$, then the Orlicz probability norm given by the function $\Theta(t) \approx t^p (\log t)^a$ (t large) is equivalent to the norm*

$$\| x \| = \left(\int_0^1 \left(1 + \log \frac{1}{t} \right)^a x^*(t)^p dt \right)^{1/p},$$

where x^* is the non-increasing rearrangement of $|x|$.

PROOF. See [1], Theorem D. □

Thus we can now deduce the following result.

THEOREM 8. *The Gaussian cotype 2 constant of the canonical embedding $l_\infty^N \to L_{2,1}^N$ is bounded by $\sqrt{\log\log N}$.*

PROOF. Let $K = \{1, 2, \ldots, N\}$, and let μ be the measure $\mu(A) = |A|/N$. Now notice that if $x \in l_\infty^N = C(K)$, $x^*(t)$ is constant over $0 \leqq t \leqq 1/N$, and hence

$$\begin{aligned}\| x \|_{L_{2,1}^N} &= \frac{1}{2}\int_0^1 \frac{x^*(t)}{\sqrt{t}}\,dt \\ &= \frac{x^*(1/N)}{\sqrt{N}} + \frac{1}{2}\int_{1/N}^1 \frac{x^*(t)}{\sqrt{t}}\,dt \\ &\leqq \left(\int_0^{1/N}\left(1+\log\frac{1}{t}\right)x^*(t)^2dt\right)^{1/2} \\ &\quad + \frac{1}{2}\left(\int_{1/N}^1 \frac{1}{t(1+\log 1/t)}\,dt\right)^{1/2}\left(\int_{1/N}^1\left(1+\log\frac{1}{t}\right)x^*(t)^2dt\right)^{1/2} \\ &\leqq c\sqrt{\log\log N}\,\| x \|_{\Phi_2}.\end{aligned}$$

This is sufficient to prove the result. □

An obvious question is the following.

PROBLEM 9. Is there a rearrangement invariant norm $\|\cdot\|_X$ on $[0, 1]$ which is strictly larger than $\|\cdot\|_{\Phi_p}$, but for which the canonical embedding $C(K) \to X(\mu)$ has Gaussian cotype p?

For $p > 2$, the answer is yes. The embedding $C(K) \to L_{p,1}(\mu)$ has cotype p (this follows from results in [2]). Hence $X = L_{\Phi_p} \cap L_{p,1}$ equipped with the norm $\| x \| = \max\{\| x \|_{\Phi_p}, \| x \|_{p,1}\}$ provides the counterexample.

For $p = 2$, the answer is no. Talagrand [10] has recently shown that if $C[0, 1] \to X$ has Gaussian cotype 2, then $\|\cdot\|_X$ is bounded by a constant times $\|\cdot\|_{\Phi_2}$.

Another problem is also suggested by Theorem 1.

PROBLEM 10. If $T: C(K) \to X$ is a linear map with Gaussian cotype 2, does it follows that there is a Radon probability measure μ on K such that $\| Tx \| \leqq c \| x \|_{L_{\Phi_2}(\mu)}$ for $x \in C(K)$?

Talagrand [10] has recently shown that this is not the case.

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